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Floquet Theory for the Stability of Boundary Layer Flows*

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A Floquet theory is presented for the stability of boundary layer flows. The spectrum of the governing differential operator is partially discrete and partially continuous, which is different from confined flows. However, no instabilities appear from the continuous part of the spectrum. © 1984 Academic Press, Inc.

1. INTRODUCTION

Much work is still being performed related to the instability of boundary layer flows; this is because of the extent of agreement between experiment and theory, even linearized theory [1]. Most theories have been concerned with a time-independent basic state. The logical next step is when the basic state may be time-periodic. The classic "Stokes layer" is the earliest example, but like plane Couette flow in the (bounded) steady case, it is not typical of its class as a whole, in that no instability is predicted by the linear theory [2]. A more representative flow has been handled recently by von Kerczek [3]. The techniques he used there are also being applied to the oscillatory asymptotic suction profile [4].

The concern of the present work is to justify a Floquet representation which most authors invoke when they study instabilities of periodic boundary layers. Furthermore, many of the modern theories of nonlinear instability use a Floquet representation as a means of understanding bifurcating periodic solutions of the equations governing instability [5, Chap. II]. For these two reasons then it is expedient to see under what conditions a Floquet representation might be valid. The governing equation is a parabolic partial differential equation, which is most con-

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veniently written as a temporally inhomogeneous periodic evolution equation

$$M \frac{d\Phi}{dt} + Q\Phi = P(t)\,\Phi, \tag{1.1}$$

where

$$P(t + 2\pi) = P(t).$$

The operators M, Q, P are differential operators in space. When P is independent of time the problem may be reduced by the normal mode approach to an ordinary differential equation eigenvalue problem. Problems similar to (1.1) have been studied under different conditions. For the purposes here, two relevant works are the dissertation by Gould [6] and the lecture notes by Henry [7]. Those authors do not consider explicitly what phenomena will occur if the spatial interval is unbounded (boundary layers). Thus their works apply most directly to "confined" oscillatory flows.

In any case it is desirable to know under what conditions solutions of (1.1) of the form

$$\Phi = e^{\lambda t}\phi \tag{1.2}$$

will exist, where ϕ is to be time-periodic of period 2π. It is proved that the existence of Floquet solutions (1.2) depend on the spectrum of the (bounded) period map $U(t)$ of (1.1). (The period map is a special type of *evolution operator* usually defined for initial value problems. A Floquet problem is a boundary value problem in time, however.) The spectrum of the period map will, in general, consist of a point spectrum, i.e., eigenvalues and a continuous or essential spectrum as in the steady case [8]. With an eye towards applications to the asymptotic suction profile, the boundary layer will be allowed to have a (steady) transverse component at infinity. So, in this work, the oscillatory behavior will be confined to the streamwise direction. This excludes problems such as that of Kelly [10], with oscillatory suction.

2. The Governing Equation

The differential operators M, Q, P in (1.1) are normally defined on a Hilbert space such as $\mathscr{L}_2[0, \infty)$. The function Φ is a map from $\mathscr{R}$ into the Hilbert space.

First the operators are given explicitly,

$$Q\Phi = q\Phi \equiv \{R^{-1}(D^2-\alpha^2)^2 + [v(y)(-D^2+\alpha^2)+v''(y)]\,D\}\,\Phi, \qquad \Phi \in \operatorname{dmn} Q, \tag{2.1}$$

$$P(t)\,\Phi = p\Phi \equiv -i\alpha[u(y,t)(-D^2+\alpha^2)+u''(y,t)]\,\Phi, \qquad \Phi \in \operatorname{dmn} P(t), \tag{2.2}$$

$$M\Phi = m\Phi \equiv (-D^2+\alpha^2)\,\Phi, \qquad \Phi \in \operatorname{dmn} M. \tag{2.3}$$

The domains are specified in what follows.

Here D and prime denote y derivatives. The constants are $\alpha > 0$ (wave number) and $R > 0$ (Reynolds number). The coefficients $u(y, t)$ and $v(y)$ are the x and y components of the basic state defined on $[0, \infty) \times (-\infty, \infty)$. As they are written, they are required to have second derivatives in y. This is not a crucial restriction since either pseudo-derivatives may be introduced or generalized functions may be used. However $u(y, t)$ must be Hölder continuous in t on any closed bounded interval. It is assumed that

$$\lim_{y\to\infty} u(y,t) = u_0 + u_1 e^{it}, \qquad -\infty < t < \infty, \tag{2.4a}$$

$$\lim_{y\to\infty} v(y) = v_0. \tag{2.4b}$$

The (finite) constants u_0, u_1, and v_0 characterize the free stream values of $u(y, t)$ and $v(y)$. By convention we take $u_0 \geqslant 0$ and consider the physically interesting case $v_0 \leqslant 0$. It is also required that

$$\lim_{y\to\infty} |u''(y,t)| = 0, \qquad -\infty < t < \infty, \tag{2.5a}$$

$$\lim_{y\to\infty} |v''(y)| = 0. \tag{2.5b}$$

Certain integrability conditions ensure the spectral resolution of the problem. These conditions, which follow, are met in all physically realistic flows. Suppose that

$$\lim_{z\to\infty} \int_z^{z+1} |u(y,t) - u_0 - u_1 e^{it}|^2\,dy = \lim_{z\to\infty} \int_z^{z+1} |u''(y,t)|^2\,dy = 0, \qquad -\infty < t < \infty, \tag{2.6a}$$

$$\lim_{z\to\infty} \int_z^{z+1} |v(y) - v_0|^2\,dy = \lim_{z\to\infty} \int_z^{z+1} |v''(y)|^2\,dy = 0, \tag{2.6b}$$

$$(u(y,t) - u_0 - u_1 e^{it}), \qquad (v(y) - v_0), \qquad u''(y,t), \qquad \text{and} \qquad v''(y) \tag{2.7}$$

are all Lebesgue integrable in y from 0 to ∞ for every t, $-\infty < t < \infty$. In fact, for most boundary layer flows the four functions in (2.7) are of negative exponential order as $y \to \infty$. For example, for the oscillatory asymptotic suction profile [9, p. 397],

$$u(y, t) = u_0(1 - e^{-y}) + u_1(1 - e^{-ky})\, e^{it},$$

where k is a complex constant ($\operatorname{Re} k > 0$) and $v(y) = v_0 < 0$.

It has been pointed out previously [8] that when $v_0 \neq 0$, the proper Hilbert space is not $\mathscr{L}_2[0, \infty)$, but a subspace of functions with sufficiently rapid decay at infinity. Thus set $V(y) = R \int^y v(\eta)/2\, d\eta$ and let the Hilbert space be

$$\mathscr{H}_0 = \{\Phi \mid e^{-V}\Phi \in \mathscr{L}_2[0, \infty)\}. \tag{2.8a}$$

The inner product on $\mathscr{H}_0$ is for $\Phi, \psi \in \mathscr{H}_0$,

$$\langle \psi, \phi \rangle = \int_0^\infty e^{-V(y)} \bar{\psi}(y)\, \Phi(y)\, dy. \tag{2.8b}$$

The operator M is given by (2.3) on its domain, when considered in $\mathscr{L}_2[0, \infty)$:

$$\begin{aligned} \operatorname{dmn} M = \{\Phi \in \mathscr{L}_2[0, \infty) \mid \Phi, \Phi' \text{ absolutely continuous},\\ m\phi \in \mathscr{L}_2[0, \infty),\ \Phi(0) = \Phi'(0) = 0\}. \end{aligned} \tag{2.9a}$$

To avoid introducing more notation than is actually needed, M denotes the same operator in $\mathscr{L}_2[0, \infty)$ and on the subspace $\mathscr{H}_0$. Moreover, M has a certain property which is noteworthy; it is positive. This is easy to show on $\mathscr{L}_2[0, \infty)$: for $\Phi \in \operatorname{dmn} M$, $\Phi \neq 0$,

$$\langle M\Phi, \Phi \rangle = \int_0^\infty [|\Phi'|^2 + \alpha^2 |\Phi|^2]\, dy > \alpha^2 \|\Phi\|^2 > 0. \tag{2.9b}$$

Though M is positive on $\mathscr{L}_2[0, \infty)$, it has no everywhere defined inverse. Nevertheless, the *generalized inverse* [14] $M^\dagger$ may be defined as the integral operator whose kernel $g^\dagger(y, \xi)$ satisfies

$$\left(-\frac{\partial^2}{\partial y^2} + \alpha^2\right) g^\dagger(y, \xi) = \delta(y - \xi) - g_J(y, \xi) \tag{2.10a}$$

$$g^\dagger(0, \xi) = \frac{\partial g^\dagger}{\partial y}(0, \xi) = 0,\ g^\dagger, \frac{\partial g^\dagger}{\partial y} \in \mathscr{L}_2[0, \infty), \tag{2.10b}$$

where $g_J(y, \xi)$ is the kernel of the projection operator onto the null space of M^*:

$$J\phi(y) = \int_0^\infty g_J(y, \xi)\, \phi(\xi)\, d\xi. \tag{2.11}$$

Now

$$M^*\phi = m\phi = (-D^2 + \alpha^2)\, \phi, \qquad \phi \in \operatorname{dmn} M^*, \tag{2.12a}$$

$$\operatorname{dmn} M^* = \{\phi \in \mathscr{L}_2[0, \infty) \mid \phi, \phi' \text{ absolutely continuous}, m\phi \in \mathscr{L}_2[0, \infty)\}. \tag{2.12b}$$

The null space of M^*, nul M^*, is spanned by $e^{-\alpha y}$, so that

$$g_J(y, \xi) = e^{-\alpha y} \left[\int_0^\infty e^{-2\alpha y}\, dy \right]^{-1} e^{-\alpha\xi} = 2\alpha e^{-\alpha(y+\xi)}.$$

The generalized inverse has the properties

$$MM^\dagger = I - J$$

and

$$M^\dagger M = I,$$

since nul M is empty. The kernel is

$$g^\dagger(y, \xi) = \frac{e^{-\alpha|y-\xi|} - e^{-\alpha(y+\xi)}}{2\alpha} - ye^{-\alpha(y+\xi)}. \tag{2.13}$$

Define

$$M\Phi = Z \tag{2.14a}$$

so that

$$\Phi = M^\dagger Z. \tag{2.14b}$$

With these definitions the evolution equation (1.1) is equivalent to

$$\frac{dZ}{dt} + AZ = B(t)\, Z, \tag{2.15a}$$

where

$$Q\Phi = QM^\dagger Z \equiv AZ, \tag{2.15b}$$

$$P(t)\, \Phi = P(t)\, M^\dagger Z \equiv B(t)\, Z. \tag{2.15c}$$

The domains of the operators in (2.15) are determined next. Suppose $f \in \text{nul } M^*$, then by (2.14)

$$\langle Z, f \rangle = \langle M\Phi, f \rangle = \langle \Phi, M^* f \rangle = 0.$$

Thus $Z \perp \text{nul } M^*$, and this is the required boundary condition for (2.15b):

$$\int_0^\infty Z(y)\, e^{-\alpha y}\, dy = 0.$$

Conversely the domain of A is defined as

$$\text{dmn } A = \left\{ Z \in \mathscr{H}_0 \mid Z, Z' \text{ absolutely continuous}, \right.$$

$$\left. qM^\dagger Z \in \mathscr{H}_0, \int_0^\infty e^{-\alpha y} Z(y)\, dy = 0 \right\}. \tag{2.16}$$

From (2.15c) and (2.2)

$$\begin{aligned} B(t)\, Z = P(t)\, M^\dagger Z &= -i\alpha[u(y, t)(-D^2 + \alpha^2) + u''(y, t)]\, M^\dagger Z \\ &= -i\alpha[u(I - J) + u'' M^\dagger]\, Z = -i\alpha[u + u'' M^\dagger]\, Z \end{aligned} \tag{2.17}$$

with (2.14b). Hence, since $M^\dagger$ is a bounded operator, $B(t)$ is a bounded operator with domain independent of t, so it is sufficient to take $\text{dmn } B(t) = \mathscr{H}_0 \cap \text{rng } M$. Note that because of the assumed conditions on u and u'' as functions of t, $B(t)$ is Hölder continuous for each t in any closed subinterval of $\mathbb{R}$. It is also important that A is a sectorial operator in the following sense.

DEFINITION 2.1 [7, p. 18]. An operator A in a Hilbert space $\mathscr{H}$ is a *sectorial* operator if it is a closed, densely defined operator such that, for some θ in $(0, \pi/2)$, for some $C \geqslant 1$ and real a, the sector

$$\Lambda_{a,\theta} = \{\lambda \mid \theta \leqq |\arg(\lambda - a)| \leqq \pi, \lambda \neq a\} \tag{2.18a}$$

is in the resolvent set of A and

$$\|(A - \lambda)^{-1}\| \leqq C/|\lambda - a| \qquad \text{for all } \lambda \in \Lambda_{a,\theta}. \tag{2.18b}$$

The explicit demonstration that A (2.15b), (2.16) is sectorial is provided in the Appendix. With that proviso, the following existence theorem is applicable to (2.15a).

THEOREM 2.2 [7, p. 190]. *Suppose A is a sectorial operator in $\mathscr{H}$ and*

$B(t)$ with dmn $B(t) =$ dmn A^{α} *is a bounded linear operator on* $\mathscr{H}$ *for each* t *in* $t_0 \leqslant t \leqslant t_1$ $(-\infty < t_0 < t_1 < \infty)$, *for some* $0 \leqslant \alpha < 1$ (dmn $A^0 = \mathscr{H}$), *and that* $t \to B(t)$, *is Hölder continuous on* $[t_0, t_1]$.

For any $\Phi_0 \in \mathscr{H}$ *there exists a unique solution* $\Phi(t) = \Phi(t; \tau, \Phi_0)$ *of* (2.15a),

$$t_0 \leqslant \tau < t \leqslant t_1, \qquad \Phi(\tau) = \Phi_0 \tag{2.19}$$

and $\Phi_0 \to \Phi(t; \tau, \Phi_0)$ *is linear and bounded in* $\mathscr{H}$, *so we write*

$$\Phi(t; \tau, \Phi_0) = T(t, \tau)\, \Phi_0, \qquad t \geqslant \tau. \tag{2.20}$$

This family of evolution operators $\{T(t, \tau), t_0 \leqslant \tau \leqslant t \leqslant t_1\}$ *has the following properties*:

(a) $T(\tau, \tau) = I$; $T(t, s)\, T(s, \tau) = T(t, \tau)$ *if* $t \geqslant s \geqslant \tau$;

(b) $\{T(t, \tau), t \geqslant \tau\}$ *is strongly continuous in* (t, τ) *with values in the space of all bounded linear operators* dmn $A^{\beta} \to$ dmn A^{β} *for any* $0 \leqslant \beta < 1$.

(*Two other conclusions are not presented here.*)

This theorem of Henry asserts the existence of a solution to the initial-value problem. The application to periodic linear systems follows in the next section.

3. Floquet Theory

The Floquet theory is concerned with solutions of (2.15a) when $B(t) = B(t + 2\pi)$ is periodic.

Definition 3.1 [7, p. 148]. The *period map* (Poincaré map) is

$$U(t) = T(t + 2\pi, t), \tag{3.1}$$

where $T(t, \tau)$ is defined in Theorem 2.2. The nonzero eigenvalues of $U(t)$ are called *characteristic multipliers.*

Besides the eigenvalues of $U(t)$ one allows for the possibility that the spectrum $\sigma(U(t))$ may have a continuous part. In particular there may be an essential spectrum.

Definition 3.2. The *resolvent set* $\rho(U(t))$ is the set of complex numbers λ such that $(U(t) - \lambda)^{-1}$ exists and is bounded. Otherwise $\lambda \in \sigma(U(t))$, the spectrum of $U(t)$. The *essential spectrum*, $\sigma_e(U(t))$, is the set of complex numbers λ (possibly depending on t) such that the range, rng$(U(t) - \lambda)$, is not closed. It is clear from Definition 3.1 that since $B(t)$ is periodic, with Theorem 2.2, $U(t+2\pi)=U(t)$ for all t.

LEMMA 3.3 [7, p. 197]. *The characteristic multipliers are independent of time, i.e., the nonzero eigenvalues of $U(t)$ coincide with those of $U(\tau)$. In fact, $\sigma(U(t))\backslash\{0\}$ is independent of t.*

We turn next to a natural alternative formulation of the problem, one which is in some ways computationally simpler.

Let L be the partial differential operator defined on $\Omega=[0,\infty)\times[0,2\pi]$ by

$$L\phi = l\phi(y,t)\equiv -m\,\partial\phi(y,t)/\partial t-(q-p)\,\phi(y,t), \tag{3.2}$$

$\phi\in\operatorname{dmn} L$, where m, q, and p are given by (2.1)–(2.3). Let $g^{\dagger}(y,\xi)$ be as defined in the last section. Define

$$\zeta = m\phi \tag{3.3a}$$

so that

$$\phi(y,t)=\int_0^\infty g^{\dagger}(y,\xi)\,\zeta(\xi,t)\,d\xi, \tag{3.3b}$$

and

$$(S\zeta)(y,t)\equiv l\int_0^\infty g^{\dagger}(y,\xi)\,\zeta(\xi,t)\,d\xi\equiv(\mathscr{s}\zeta)(y,t) \tag{3.4}$$

$$=-\frac{\partial\zeta}{\partial t}(y,t)-\int_0^\infty (q-p)\,g^{\dagger}(y,\xi)\,\zeta(\xi,t)\,d\xi,\qquad \zeta\in\operatorname{dmn} S.$$

The domain of S can be specified when the proper Hilbert space setting is introduced. Analogous to (2.8a) define

$$\mathscr{H}'_0(\Omega)=\{\phi\mid e^{-V(y)}\phi\in\mathscr{L}_2(\Omega)\} \tag{3.5a}$$

and take the inner product to be for $\phi,\psi\in\mathscr{H}'_0$:

$$(\phi,\psi)=\int_0^{2\pi}dt\int_0^\infty e^{-2V(y)}\phi(y,t)\,\bar{\psi}(y,t)\,dy. \tag{3.5b}$$

Thus the domain of S is

$$\operatorname{dmn} S=\{\zeta\in\mathscr{H}'_0(\Omega)\mid \zeta,\,\partial\zeta/\partial y \text{ absolutely continuous}, \mathscr{s}\zeta\in\mathscr{H}_0\} \tag{3.6}$$

The next objective is to relate the two formulations of the problem. (It has not been explicitly demonstrated here but the period map $U(t)$ has the representation

$$U(t)\,\Phi(y,t)=\int_0^\infty F(y,t+2\pi,\xi,t)\,\Phi(\xi,t)\,d\xi, \tag{3.7}$$

where $F(y, t, \xi, \tau)$ is the Green's function for the initial-value problem for (3.2) [6].)

LEMMA 3.4 [Cf. 6, p. 47]. *A complex number $\mu \neq 0$ is an eigenvalue of $U(0)$ if and only if the equation*

$$L\phi=\lambda M\phi \tag{3.8}$$

has a nonzero (2π-periodic) solution, where $\mu=e^{2\pi\lambda}$. That is, $\mu\in\sigma_p(U(0))$, μ is in the point spectrum of $U(0)$, if and only if $\lambda\in\sigma_p(L, M)$, λ is in the point M-spectrum of L.

Proof. From Theorem 2.2 and Definition 3.1, for all $\Phi\in \text{dmn}\, U(t)$,

$$\Phi(y,t+2\pi)=U(t)\,\Phi(y,t). \tag{3.9}$$

If $\mu\in\sigma_p(U(0))$, then for some $\Phi\in\text{dmn}\,U(t)$,

$$U(0)\,\Phi(y,0)=\Phi(y,2\pi)=\mu\Phi(y,0), \tag{3.10}$$

and $l\Phi=0$.

Let ϕ be a solution of (3.8). Set $\Phi(y,t)=e^{\lambda t}\phi(y,t)$; then

$$l\Phi=-\lambda e^{\lambda t}(m\phi)+e^{\lambda t}(l\phi)=e^{\lambda t}(-\lambda m\phi+l\phi)=0. \tag{3.11}$$

Also, since ϕ is a solution of (3.8), ϕ is periodic while

$$\mu\Phi(y,0)=\mu\phi(y,0)=\mu\phi(y,2\pi)=\mu e^{-2\pi\lambda}\Phi(y,2\pi)=\Phi(y,2\pi), \tag{3.12}$$

so that μ is an eigenvalue of $U(0)$.

Conversely, let μ be an eigenvalue of $U(0)$. Then there exists a $\Phi(y,t)$ satisfying $l\Phi=0$ and $\mu\Phi(y,0)=\Phi(y,2\pi)$. Now define

$$\phi(y,t)=e^{-\lambda t}\Phi(y,t), \qquad \text{where} \quad e^{2\pi\lambda}=\mu.$$

Then

$$l\phi=\lambda e^{-\lambda t}m\Phi+e^{-\lambda t}(l\Phi),$$

so

$$l\phi-\lambda e^{-\lambda t}m\Phi=e^{-\lambda t}l\Phi$$

or

$$(l - \lambda m)\,\phi = 0.$$

Also $\phi(y, t)$ is periodic in t:

$$\phi(y, 0) = \Phi(y, 0) = \frac{1}{\mu}\,\Phi(y, 2\pi) = e^{2\pi\lambda} e^{-2\pi\lambda} \phi(y, 2\pi) = \phi(y, 2\pi). \quad (3.13)$$

Hence $L\phi = \lambda M\phi$. ∎

DEFINITION 3.5 [Cf. 9, p. 1393]. The *essential M-spectrum* of L, $\sigma_e(L, M)$ is the set of complex numbers λ such that $\operatorname{rng}(L - \lambda M)$ is not closed. For the formal differential operators l and m in (3.2) and (2.3), $\sigma_e(l, m)$ is the essential spectrum of the minimal operator corresponding to $l - \lambda m$.

LEMMA 3.6. *A complex number $\lambda \in \sigma_e(l, m)$, if and only if $0 \in \sigma_e(l, m)$.*

Proof. By Definition 3.4, if $0 \in \sigma_e(l, m)$, the $\operatorname{rng} l$ is not closed. Let $\{\phi_n\} \in \operatorname{dmn} l \subset \operatorname{dmn} m$, such that $\phi_n \to \phi$, and suppose that for some λ, $\operatorname{rng}(l - \lambda m)$ is closed; then $(l - \lambda m)\,\phi_n \to (l - \lambda m)\,\phi$. Define $\Phi_n(y, t) = e^{\lambda t}\phi_n(y, t) \to e^{\lambda t}\phi(y, t) \equiv \Phi(y, t)$. Then $l\Phi_n = -\lambda e^{\lambda t} m\phi_n + e^{\lambda t} l\phi_n = e^{\lambda t}(l - \lambda m)\,\phi_n \to e^{\lambda t}(l - \lambda m)\,\phi = l\Phi$. Thus l has closed range.

Suppose on the other hand that l has closed range. Consider $\Phi_n \in \operatorname{dmn} l \subset \operatorname{dmn} m$ such that $\Phi_n \to \Phi$. Define $\phi_n(y, t) = e^{-\lambda t}\Phi_n(y, t) \to e^{-\lambda t}\Phi \equiv \phi(y, t)$. Then $(l - \lambda m)\,\phi_n = e^{-\lambda t} l\Phi_n \to e^{-\lambda t} l\Phi = (l - \lambda m)\,\phi$, since l has closed range. Thus $\operatorname{rng}(l - \lambda m)$ is closed.

If neither $\operatorname{rng} l$ nor $\operatorname{rng}(l - \lambda m)$ is closed neither is the other. Thus if $\sigma_e(l, m)$ is not empty, $0 \in \sigma_e(l, m)$. ∎

LEMMA 3.7. *A complex number $\lambda \in \sigma(L, M)$ if and only if $\mu \in \sigma(U(t))$, where $\mu = e^{2\pi\lambda}$, $\mu \neq 0$. Furthermore, $\lambda \in \sigma_e(L, M)$ if and only if $\mu \in \sigma_e(U(t))$.*

Proof. If problem (3.10) is restated as

$$M\frac{d\phi}{dt} + (Q - P(t))\,\phi + \lambda M\phi = 0, \quad (3.14)$$

with $P(t + 2\pi) = P(t)$, then as (1.1) is transformed to (2.15a) to (3.14) is transformed to

$$\frac{d\zeta}{dt} + (A - B(t))\,\zeta + \lambda\zeta = 0, \quad (3.15)$$

where $B(t + 2\pi) = B(t)$.

The Cauchy problem for (3.17) is *uniformly correct* [12, p. 192] if

(a) it has a unique solution ($T_\lambda(t, \tau)$ is invertible),

(b) $\zeta(t)$ and $\zeta'(t)$ are continuous,

(c) $T_\lambda(t, \tau)$ has closed range, where $T_\lambda(t, \tau)$ is the evolution operator for (3.15).

It is also true that [12, p. 194] the Cauchy problem for (2.15a) is uniformly correct if and only if the Cauchy problem for (3.15) is uniformly correct.

The purpose of the transformations is that if $Z = e^{\lambda t}\zeta$ then the period map for (3.15) is

$$U_\lambda(t) = T_\lambda(t+2\pi, t) = e^{-2\pi\lambda} T(t+2\pi, t) = e^{-2\pi\lambda} U(t). \tag{3.16}$$

(The existence of a periodic solution of (3.15), $\zeta(t+2\pi) = \zeta(t)$, means that $1 \in \sigma_p(U_\lambda(t))$.) Suppose $\mu \in \rho(U(t))$. Then from (3.16),

$$(U(t) - \mu)^{-1} = e^{-2\pi\lambda}(U_\lambda(t) - \mu e^{-2\pi\lambda})^{-1} \tag{3.17}$$

and $\mu e^{-2\pi\lambda} \in \rho(U_\lambda(t))$. The converse is obviously true. Consequently $\mu \in \sigma(U(t))$ if and only if $\mu e^{-2\pi\lambda} \in \sigma(U_\lambda(t))$. The Cauchy problem for (3.15) is not uniformly correct if $\lambda \in \sigma(L, M)$. Condition (a) means $\lambda \notin \sigma(L, M)$ if and only if $\mu \notin \sigma_e(U(t))$. From Definition 3.4 condition (c) means $\lambda \notin \sigma_e(L, M)$ if and only if $\mu \notin \sigma_e(U(t))$. ▮

The identifications made in Lemmas 3.3 and 3.7 indicate that there is a 1:1 correspondence between the spectral points of $U(t)$ and those of (L, M). This is important because most of the proven results concern the spectrum of $U(t)$ while the spectrum $\sigma(L, M)$ can be more easily determined.

DEFINITION 3.8 [7, p. 30]. If A is an operator with domain and range in a Hilbert space $\mathscr{H}$, and $\sigma(A)$ denotes the spectrum, a set $\sigma_s = \sigma(A) \cup \{\infty\} \equiv \hat{\sigma}(A)$ is a *spectral set* if both σ_s and $\hat{\sigma}(A)\backslash\sigma$ are closed in the extended plane $\mathbb{C} \cup \{\infty\}$.

THEOREM 3.9 [7, p. 198]. *Suppose σ_1 is a spectral set for $\sigma(U(t))$ for all t; the usual case is when σ_1 is a finite collection of isolated eigenvalues, or the complement of such a set. Then for each t, the space $\mathscr{H}$ may be decomposed as $\mathscr{H} = \mathscr{H}_1(t) \oplus \mathscr{H}_2(t)$, the direct sum of closed subspaces invariant under $U(t)$. $\sigma(U(t)|_{\mathscr{H}_1(t)}) = \sigma_1$, $\sigma(U(t)|_{\mathscr{H}_2(t)}) = \sigma(U(t))\backslash\sigma_1$. If $t \geqslant \tau$, $T(t, \tau)$ maps $\mathscr{H}_1(\tau)$ into $\mathscr{H}_1(t)$, and is a 1:1 map onto $\mathscr{H}_1(t)$ if $0 \notin \sigma_1$.*

Let $e^{2\pi\beta} = \sup\{|\mu|, \mu \in \sigma_1\}$; *then for any* $\varepsilon > 0$, *there exists* $K_\varepsilon > 0$ *such that*

$$\|T(t, \tau)\, \Phi\| \leqslant K_\varepsilon e^{(\beta+\varepsilon)(t-\tau)}\, \|\Phi\|,$$

for $t \geqslant \tau$ *and* $\Phi \in \mathscr{H}_1(\tau)$.

Now suppose $0 \notin \sigma_1$, *and let* $e^{2\pi\gamma} = \inf\{|\mu|, \mu \in \sigma_1\} > 0$. *Then* $T(t, \tau)\, \Phi$, $\Phi \in \mathscr{H}_1(\tau)$, *may be defined also for* $t \leqslant \tau$, *still satisfying* (a) *of Theorem* 2.2, *and for* $\Phi \in \mathscr{H}_1(\tau)$, *and* $\varepsilon > 0$ *and sufficiently small,*

$$\|T(t, \tau)\, \Phi\| \leqslant K_\varepsilon e^{(\gamma-\varepsilon)(t-\tau)}\, \|\Phi\|, \qquad t \leqslant \tau.$$

In fact, provided there is a path Γ *in the complex plane, disjoint from* σ_1, *joining* 0 *to* ∞, *we have a kind of* Floquet representation. *There exists a family of bounded invertible operators* $E(t)$: $\mathscr{H}_1(\tau_0) \to \mathscr{H}_1(t)$ *for all* t *with* $E(t + 2\pi) = E(t)$, $E(\tau_0) = I$, *and a bounded operator* C *on* $\mathscr{H}_1(\tau_0)$ *with spectrum* $\sigma(C) = (1/2\pi) \ln \sigma_1$ *and for* $\Phi \in \mathscr{H}_1(\tau)$ *and all* t, τ,

$$T(t, \tau)\, \Phi = E(t)\, e^{C(t-\tau)} E^{-1}(\tau)\, \Phi.$$

4. Perturbation of Spectral Operators—The Free Stream Problem

With conditions (2.4)–(2.7) one is led to consider (3.14) in the form

$$M \frac{d\phi}{dt} + (Q_0 - P_0(t))\, \phi + (Q_1 - P_1(t))\, \phi + \lambda M\phi = 0, \tag{4.1a}$$

where

$$Q_0\phi = q_0\phi \equiv [R^{-1}(-D^2 + \alpha^2)^2 + v_0(-D^2 + \alpha^2)\, D]\, \phi,\ \phi \in \operatorname{dmn} Q, \tag{4.1b}$$

$$P_0(t)\, \phi = p_0\phi \equiv -i\alpha[(u_0 + u_1 e^{it})(-D^2 + \alpha^2)]\, \phi, \qquad \phi \in \operatorname{dmn} P(t), \tag{4.1c}$$

and

$$Q_1 = Q - Q_0, \qquad P_1(t) = P(t) - P_0(t). \tag{4.1d}$$

The observation of interest is that the operator $Q - P(t)$ is a *relatively compact* perturbation of $Q_0 - P_0(t)$ [8]. This kind of perturbation leaves the essential spectrum of the operator unchanged and hence the determination of the essential Floquet spectrum is rendered more simple. The actual computation is made by determining the spectrum for the "free stream" problem

$$S_1\zeta = \lambda\zeta, \tag{4.2a}$$

where from (3.4)

$$(S_1\zeta)(y,t) = -\frac{\partial\zeta}{\partial t}(y,t) - \int_0^\infty (q_0 - p_0)\, g^\dagger(y,\xi)\, \zeta(\xi,t)\, d\xi,$$
$$\zeta \in \text{dmn } S_1 = \text{dmn } S. \tag{4.2b}$$

The computation here is based on a knowledge of the steady case so that (4.2a) is obtained from (see Appendix Eq. A.2)

$$\frac{d\zeta}{dt} + (A_0 - B_0(t))\,\zeta + \lambda\zeta = 0 \tag{4.3}$$

as

$$\frac{d\zeta}{dt} + [R^{-1}(-D^2+\alpha^2) + v_0 D + i\alpha(u_0 + u_1 e^{it})]\,\zeta + \lambda\zeta = 0. \tag{4.4}$$

It is useful to define the operator N, where dmn N = dmn A and

$$N\zeta \equiv [R^{-1}(-D^2+\alpha^2) + v_0 D + i\alpha u_0]\,\zeta, \qquad \zeta \in \text{dmn } N. \tag{4.5}$$

The Green's function $G(y, \xi, t, \tau; \lambda)$ for (4.4) is known to satisfy [13, p. 282]

$$\frac{dG}{dt} + (i\alpha u_1 e^{it} + N + \lambda)\, G = -\delta(t-\tau)\,\delta(y-\xi), \tag{4.6a}$$

with

$$G(y, \xi, 0, \tau; \lambda) = G(y, \xi, 2\pi, \tau; \lambda). \tag{4.6b}$$

The operator N, whose spectral resolution is known [8, Sect. 2(c)] is taken as constant in this calculation. The kernel is

$$G(y, \xi, t, \tau; \lambda) = K(t, \tau; \lambda)[e^{-2\pi(N+\lambda)}\eta(\tau - t) + \eta(t-\tau)]\,\delta(y-\xi), \tag{4.7a}$$

where

$$K(t, \tau; \lambda) = \{\exp[\alpha u_1(e^{i\tau} - e^{it}) + (N+\lambda)(t-\tau)]\}/(1 - e^{-2\pi(N+\lambda)}) \tag{4.7b}$$

and

$$\begin{aligned} \eta(t-\tau) &= 1, \qquad t > \tau, \\ &= 0, \qquad t < \tau. \end{aligned} \tag{4.7c}$$

The spectral representation for N is

$$\delta(y-\xi) = -\frac{1}{2\pi i}\oint_{\Gamma'} h(y, \xi; \gamma)\, e^{Rv_0(y-\xi)/2}\, d\gamma, \tag{4.8}$$

where Γ' is a contour enclosing the branch cut in $h(y, \xi; \gamma)$, the Green's function for the resolvent of N [8, Eq. (2.39b)]. This branch cut is the continuous (essential) spectrum of N; in fact the spectrum of N consists only of a continuous part. Thus if in (4.8) one sets

$$\gamma = R^{-1}(\omega^2 + \alpha^2 + R^2 v_0^2/4 + i\alpha R u_0), \qquad 0 \leqslant \omega < \infty, \tag{4.9}$$

the spectrum of N is swept out [8, Eq. (2.36)]. Then the kernel of the resolvent of S_1 [13, p. 286] is

$$G(y, \xi, t, \tau; \lambda) = -\frac{\exp[\alpha u_1(e^{i\tau} - e^{it})]}{2\pi i}\oint_{\Gamma'} \frac{e^{(\gamma+\lambda)(t-\tau)}}{1 - e^{-2\pi(\gamma+\lambda)}}$$
$$\times [e^{-2\pi(\gamma+\lambda)}\eta(\tau - t) + \eta(t-\tau)]\, h(y, \xi; \gamma)\, d\gamma, \tag{4.10}$$

since N is a *spectral operator* [8, Theorem 2]. The spectrum is given by the branch poles in the denominator, where $-2\pi(\gamma+\lambda) = 2n\pi i$ and n is any integer. Hence the spectrum of S_1 is the union of a denumerably infinite set of parallel halflines given parametrically by

$$\lambda_n = -\gamma - ni = -R^{-1}(\omega^2 + \alpha^2 + R^2 v_0^2/4 + i\alpha R u_0 + Rni),$$
$$n = 0, \pm 1, \pm 2, \ldots, 0 \leqslant \omega < \infty. \tag{4.11}$$

From Definition 3.8, it follows that each of the halflines in (4.11) is related to a spectral set for N, that is $\sigma_{s_n} = e^{2\pi\lambda_n}$. This is because of the fact that N is a spectral operator. Furthermore, $0 \notin \sigma_{s_n}$ for any n, so a path from 0 to ∞ may be chosen disjoint from each of the σ_{s_n}. In the light of Theorem 3.9, we have a kind of Floquet representation. This is of great interest theoretically, and relates to completeness. Of practical interest is the point in the Floquet spectrum farthest to the right. According to the definition of the Floquet parameter in (1.2), the rays (4.11) all lie in the "stable" half of the complex plane, $\operatorname{Re} \lambda < 0$. The spectrum of L will consist of a continuous (essential) spectrum, identical to that of S_1, and a point spectrum, i.e., eigenvalues from which any instabilities in the periodic flow can be determined.

5. Concluding Remarks

The Floquet representation which is frequently used for confined flows is justified for boundary layer flows as well. In the case of boundary layers the

Floquet spectrum is part continuous and part discrete. However all instabilities are associated with the discrete part which can only be determined by the complete profile and only accurately with the use of a computer as well. The techniques for these types of calculations are presented elsewhere [3].

Other flows which would fall under this type of analysis would be oscillatory jets and shear layers and the oscillatory Ekman layer. This last flow is of particular interest because like the oscillatory asymptotic suction profile it satisfies the Navier–Stokes equations.

APPENDIX: THE OPERATOR A, (2.15b)–(2.16) IS SECTORIAL

Explicit calculations are to be made in this section. To facilitate this, the inner product used is

$$\langle \Psi, \Phi \rangle = \int_0^\infty e^{-Rv_0 y/2} \bar{\Psi}(y)\, \Phi(y)\, dy, \tag{A.1a}$$

for $\Psi, \Phi \in \mathscr{H}_0$. This differs from (2.8b), but has the advantage that all calculations are explicit. However, from the assumed conditions on $v(y)$ (A.1a) converges if and only if (2.8b) converges. Of course, if $v(y) \equiv v_0$, then the two integrals are the same. The whole theory may also be performed in a space $\tilde{\mathscr{H}}_0$ with inner product (A.1a) and such that

$$\tilde{\mathscr{H}}_0 = \{\Phi \mid e^{-Rv_0 y/2} \Phi \in \mathscr{L}_2[0, \infty)\}. \tag{A.1b}$$

Then $\mathscr{H}_0$ and $\tilde{\mathscr{H}}_0$ are virtually equivalent. For practical purposes one or the other may be preferred. The operator A is given explicitly as

$$AZ = \{R^{-1}(-D^2+\alpha^2)^2 + [v(y)(-D^2+\alpha^2) + v''(y)]\, D\}\, M^\dagger Z, \quad Z \in \operatorname{dmn} A. \tag{A.2}$$

The operator will be written as the sum of two operators $A = A_0 + A_1$, where A_1 is a small perturbation of a sectorial operator A_0.

LEMMA A.1. *Let A_0 be the operator such that*

$$\begin{aligned} A_0 Z &= \{R^{-1}(-D^2+\alpha^2)^2 + v_0(-D^2+\alpha^2)\, D\}\, M^\dagger Z \\ &= \{R^{-1}(-D^2+\alpha^2) + v_0 D\}\, Z, \qquad Z \in \operatorname{dmn} A_0 = \operatorname{dmn} A. \end{aligned} \tag{A.3}$$

Then A_0 is sectorial in $\mathscr{H} \equiv \operatorname{rng} M \cap \mathscr{H}_0 = \operatorname{rng} M$. (Note that if $\alpha > Rv_0/2$, $\operatorname{rng} M = \mathscr{H}_0$ [8].)

Proof. It will be shown that A_0 satisfies the conditions of Definition 2.1 in $\mathscr{H}$. It is clear that A_0 is closed in $\mathscr{H}$. The domain of A_0 (2.16) is dense in rng M since dmn A_0 is dense in the orthogonal complement of nul M^*.

Next, it must be shown that (2.18) holds for A_0. The construction of the resolvent $(A_0 - \lambda)^{-1}$ has been performed elsewhere [8] and in the present notation is given by

$$(A_0 - \lambda)^{-1} f = \int_0^\infty h(y, \xi; \lambda)\, e^{Rv_0(y-\xi)/2} f(\xi)\, d\xi \tag{A.4}$$

for $f \in$ rng M, with

$$h(y, \xi; \lambda) = R\left[\frac{e^{-r|y-\xi|}}{2r} + \frac{(r+v)\, e^{-r(y+\xi)}}{2r(r-v)} - \frac{e^{-ry} e^{-v\xi}}{r-v}\right], \tag{A.5a}$$

where

$$v = -Rv_0/2 + \alpha \tag{A.5b}$$

and

$$r = +\sqrt{R(\alpha^2/R + Rv_0^2/4 - \lambda)} \tag{A.5c}$$

is the positive square root. The kernel of the operator can have poles only at $r = 0$ and $r = v$. However a consideration of the limits $\lim_{r \to 0} h(y, \xi; \lambda)$ and $\lim_{r \to v} h(y, \xi; \lambda)$ shows these limits to be finite. Thus, the resolvent has no poles and A_0 has no eigenvalues. Hence $\sigma_p(A_0)$ is empty. There is a continuous spectrum $\sigma_c(A_0) = \{\lambda \in \mathbb{R} \mid \lambda \geqq \alpha^2/R + Rv_0^2/4\}$, lying along a portion of the real line since h has a branch point at $\lambda = \alpha^2/R + Rv_0^2/4$. Thus in (2.18a) it suffices to take $a = \alpha^2/R + Rv_0^2/4$ and θ any value in $(0, \pi/2)$.

Finally, it is shown that (2.18b) holds. For $f \in mgM$ and $\lambda \in \rho(A_0)$, from (A.4) and (A.5a),

$$(A_0 - \lambda)^{-1} f = Re^{Rv_0 y/2} \int_0^\infty e^{-Rv_0\xi/2} \left[\frac{e^{-r|y-\xi|}}{2r} + \frac{(r+v)\, e^{-r(y+\xi)}}{2r(r-v)} - \frac{e^{-ry} e^{-v\xi}}{r-v}\right] f(\xi)\, d\xi. \tag{A.6}$$

Since $f \in$ rng M, $f \perp$ nul M^*, so the third term vanishes after integration over ξ. Moreover, we have the bound

$$\|(A_0 - \lambda)^{-1}\| = \sup_{f \in \text{rng} M} \frac{\|(A_0 - \lambda)^{-1} f\|}{\|f\|}. \tag{A.7}$$

Define

$$h_1(y, \xi; \lambda) = \frac{Re^{-r|y-\xi|}}{2r} \tag{A.8a}$$

$$h_2(y, \xi; \lambda) = R\left[\frac{(r+v)\, e^{-r(y+\xi)}}{2r(r-v)} - \frac{e^{-ry}e^{-v\xi}}{r-v}\right]. \tag{A.8b}$$

Thus $h = h_1 + h_2$ consists of two parts. The first of which h_1, is not a Hilbert–Schmidt kernel, even for $Re r > 0$. However,

$$\int_0^\infty |h_1(y, \xi; \lambda)|\, dy = \frac{R(2 - e^{-(r+\bar{r})\xi/2})}{|r|(r+\bar{r})} \leqq \frac{2R}{|r|(r+\bar{r})}$$

$$\leqq \frac{\operatorname{cosec}(\theta/2)}{|\lambda - a|} = \frac{C_1}{|\lambda - a|}, \tag{A.9a}$$

$\xi \in [0, \infty)$, $\lambda \in \Lambda_{a,\theta}$. Similarly,

$$\int_0^\infty |h_1(y, \xi; \lambda)|\, d\xi \leqq \frac{C_1}{|\lambda - a|}, \tag{A.9b}$$

$y \in [0, \infty)$, $\lambda \in \Lambda_{a,\theta}$. Thus,

$$\left\| \int_0^\infty h_1(y, \xi; \lambda)\, e^{Rv_0(y-\xi)/2} f(\xi)\, d\xi \right\|$$

$$= \left\{ \int_0^\infty dy\, e^{-v_0 y} \left| \int_0^\infty h_1(y, \xi; \lambda)\, e^{Rv_0(y-\xi)/2} f(\xi)\, d\xi \right|^2 \right\}^{1/2}$$

$$= \left\{ \int_0^\infty dy \int_0^\infty h_1(y, \xi; \lambda)\, f(\xi)\, e^{-Rv_0\xi/2} \int_0^\infty \overline{h_1(y, z; \lambda)}\, \overline{f(z)}\, e^{-Rv_0 z/2}\, dz \right\}^{1/2}$$

$$\leqq \|f\| \left(\frac{C_1}{|\lambda - a|} \right). \tag{A.10}$$

The last inequality follows from Fubini's theorem and (A.9). Though h_1 was not, h_2 is a Hilbert–Schmidt kernel, when $v > 0$ and $Re r > 0$. If $v \leqq 0$, h_2 is not Hilbert–Schmidt, but the second term in h_2 may be ignored since (from the assumed conditions on f) after integration over ξ, its contribution to (A.6) vanishes, while the first term in h_2 is amenable to the analysis to follow. When $v \leqq 0$; let

$$\tilde{h}_2(y, \xi; \lambda) = \frac{(r+v)}{2r(r-v)}\, e^{-r(y+\xi)}. \tag{A.11}$$

Then

$$\left\| \int_0^\infty h_2(y, \xi; \lambda)\, e^{Rv_0(y-\xi)} f(\xi)\, d\xi \right\|$$
$$= \left\| \int_0^\infty \tilde{h}_2(y, \xi; \lambda)\, e^{Rv_0(y-\xi)/2} f(\xi)\, d\xi \right\|$$
$$= \left\{ \int_0^\infty dy\, e^{-Rv_0 y} \left| \int_0^\infty \tilde{h}_2(y, \xi; \lambda)\, e^{Rv_0(y-\xi)/2} f(\xi)\, d\xi \right|^2 \right\}^{1/2}$$
$$\leqq \sqrt{\int_0^\infty |f(\xi)|^2 e^{-Rv_0 \xi}\, d\xi} \sqrt{\int_0^\infty \int_0^\infty |\tilde{h}_2(y, \xi; \lambda)|^2\, dy\, d\xi}$$

(by Schwarz's inequality and Fubini's theorem)

$$= \|f\| \left(\frac{R}{2\,|r|(r+\bar{r})} \left| \frac{r+v}{r-v} \right| \right) \leqq \|f\| \left(\frac{C_2}{|\lambda - a|} \right), \tag{A.12}$$

for $\lambda \in \Lambda_{a,\theta}$, $v \leqq 0$. The case $v > 0$ is not troublesome since we have explicitly

$$\sqrt{\int_0^\infty \int_0^\infty |h_2(y, \xi; \lambda)|^2\, dy\, d\xi}$$
$$= \frac{R}{2\,|r|(r+\bar{r})\,|r-v|} \left[\frac{|r+v|^2}{2} + \frac{(r+\bar{r})}{v} |r|^2 - (r+\bar{r})^2 \right]^{1/2}$$
$$\to \sqrt{5}\, R/4v^2 \quad \text{as} \quad r \to v, \text{ when } \lambda(v) \in \Lambda_{a,\theta}. \tag{A.13a}$$

Consequently,

$$\sqrt{\int_0^\infty \int_0^\infty |h_2(y, \xi; \lambda)|^2\, dy\, d\xi} \leqq \frac{C_3}{|\lambda - a|}, \tag{A.13b}$$

for $\lambda \in \Lambda_{a,\theta}$, $v > 0$. Together conditions (A.10), (A.12), (A.13b) give

$$\|(A_0 - \lambda)^{-1}\| \leqq \frac{C}{|\lambda - a|}. \quad \blacksquare$$

That $A = A_0 + A_1$ is sectorial is a consequence of Theorem A.2.

THEOREM A.2 [7, p. 19]. *Suppose* A_0 *is a sectorial operator and* $\|A_0(\lambda - A_0)^{-1}\| \leqq C'$ *for* $|\arg \lambda| \geqq \theta_0$, $|\lambda| \geqq \mu_0$ *for some positive constants*

μ_0, C', and $\theta_0 < \pi/2$. Suppose also A_1 is a linear operator with dmn $A_1 \supset$ dmn A_0,

$$\|A_1 \Phi\| \leqq \varepsilon \|A_0 \Phi\| + \kappa \|\Phi\|, \tag{A.14}$$

for all $\phi \in$ dmn A_0; ε and κ are positive constants with $\varepsilon C' < 1$. Then $A_0 + A_1$ is sectorial.

When (A.14) holds A_1 is a relatively bounded with respect to A_0, A_0-bounded [11, p. 190], we have explicitly that

$$A_1 Z = [(v(y) - V_0) D + v''(y) DM^{\dagger}] Z, \qquad Z \in \text{dmn } A. \tag{A.15}$$

The term $v''(y) DM^{\dagger}$ represents a bounded operator which is A_0-bounded with $\varepsilon = 0$. The term $(v(y) - v_0) D$ is A_0-bounded for every $\varepsilon > 0$ [11, p. 192]. Thus (A.14) will hold and $\varepsilon C' < 1$ is true if C' exists. The existence of C' follows from noting that

$$\begin{aligned} \|A_0(A_0 - \lambda)^{-1} f\| &= \|[I + \lambda(A_0 - \lambda)^{-1}] f\| \leqq \|f\| + |\lambda| \, \|(A_0 - \lambda)^{-1} f\| \\ &\leqq \left(1 + \frac{C|\lambda|}{|\lambda - a|}\right) \|f\|, \end{aligned} \tag{A.16}$$

based on previous calculations. Thus for $|\lambda|$ sufficiently large, $|\arg \lambda| \geqq \theta_0$, $\|A_0(\lambda - A_0)^{-1}\| \leqq C'$, and $A = A_0 + A_1$ is a sectorial operator.

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